Logarithmic equations

1) Solve the equations:

a) $\log_3(2x+3)=2$

b) $\log_4(3x+4)=3$

c) $\log\sqrt{3x+1}=\frac{1}{2}$

Solution:

a) $\log_3(2x+3)=2 \longrightarrow$ Use definition: $\log_A B=\otimes \Leftrightarrow B=A^{\otimes}$

So :

$$2x+3=3^2$$
$$2x+3=9$$
$$2x=6$$
$$x=3$$

condition:
$$2x+3>0$$
$$2x>-3$$
$$x>-\frac{3}{2}$$

Because $3>-\frac{3}{2}$, solution x = 3 is "good''

b) $\log_4(3x+4)=3 \longrightarrow$ again by definition…

$$3x+4=4^3$$
$$3x+4=64$$
$$3x=60$$
$$x=20$$

condition:
$$3x+4>0$$
$$3x>-4$$
$$x>-\frac{4}{3}$$

Solution satisfies the condition!

c) $\log\sqrt{3x+1}=\frac{1}{2}$

$$\log_{10}\sqrt{3x+1}=\frac{1}{2}$$
$$\sqrt{3x+1}=10^{\frac{1}{2}}$$
$$\sqrt{3x+1}=\sqrt{10}......./()^2$$
$$3x=9$$
$$x=3$$

condition:
$$\sqrt{3x+1}>0$$
$$3x+1>0$$
$$x>-\frac{1}{3}$$

So, $3>-\frac{1}{3}$, solution x = 3 is "good''

2) Solve the equations:

a) $\log_2(x-1)+\log_2(x+2)=2$

b) $\log(x^2+19)-\log(x-8)=2$

c) $\log(5-x)+2\log\sqrt{3-x}=1$

Solution:

We use : $\log_a x+\log_a y=\log_a(xy)$

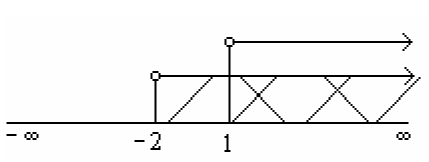
a)

$$\log_2(x-1)+\log_2(x+2)=2$$
$$\log_2(x-1)(x+2)=2 \rightarrow \text{ Conditions: } x-1>0 \text{ and } x+2>0$$
$$x>1 \quad \text{and} \quad x>-2$$

by definition: $\log_A B=\otimes \Leftrightarrow B=A^{\otimes}$

$$(x-1)(x+2)=2^2$$
$$x^2+2x-x-2=4$$
$$x^2+x-6=0$$
$$x_{1,2}=\frac{-1\pm 5}{2}$$
$$x_1=2$$
$$x_2=-3$$

Are the solutions satisfy conditions?: $x>1$ i $x>-2$



$x\in(1,\infty)$

$x_1=2$ → **'good'**

$x_2=-3$ → ‘ not good’

So, the only solution is: $x=2$

b) $\log(x^2+19)-\log(x-8)=2$

$$\log_{10}(x^2+19)-\log_{10}(x-8)=2$$

We know that: $\log_a x-\log_a y=\log_a \frac{x}{y}$

$$\log_{10}\frac{x^2+19}{x-8}=2 \quad \text{with condition: } x^2+19>0 \text{ i } x-8>0$$

$$x>8$$

$$\frac{x^2+19}{x-8}=10^2$$

$$\frac{x^2+19}{x-8}=100$$

$$x^2+19=100(x-8)$$

$$x^2+19=100x-800$$

$$x^2-100x+819=0$$

$$x_{1,2}=\frac{100\pm 82}{2}$$

$$x_1=91$$

$$x_2=9$$

Both solutions are "good" because they are more than 8.

c) $\log(5-x)+2\log\sqrt{3-x}=1$

Conditions are:

$5-x>0$		$3-x>0$
$-x>-5$	and	$-x>-3$
$x<5$		$x<3$

So: $x<3$

$$(5-x)(3-x)=10^1$$

$$15-5x-3x+x^2-10=0$$

$$x^2-8x+5=0$$

$$x_{1,2}=\frac{8\pm\sqrt{44}}{2}=\frac{8\pm 2\sqrt{11}}{2}=\frac{2(4\pm\sqrt{11})}{2}=4\pm\sqrt{11}$$

$$x_1=4+\sqrt{11}\approx 7{,}32$$

$$x_2=4-\sqrt{11}\approx 0{,}68$$

$x_1=4+\sqrt{11}$ does not meet the requirement, and the only solution is $x=4-\sqrt{11}$

3) **Solve the equations:**

a) $\log^2 x - 3\log x + 2 = 0$

b) $\log_2 x + \log_x = \frac{5}{2}$

Solution:

a) Replacement: $\log x = t$ with condition: $x > 0$

$$\log^2 x - 3\log x + 2 = 0$$
$$t^2 - 3t + 2 = 0$$
$$t_{1,2} = \frac{3 \pm 1}{2}$$
$$t_1 = 2$$
$$t_2 = 1$$

Back in the replacement : $\log_{10} x = 2$, $x = 10^2$, $x = 100$ and $\log_{10} x = 1$, $x = 10^1$, $x = 10$

b) $\log_2 x + \log_x = \frac{5}{2}$ Use: $\log_a b = \frac{1}{\log_b a}$

$\log_2 x + \frac{1}{\log_b a} = \frac{5}{2} \rightarrow$ Replacement : $\log_2 x = t$ with condition: $x > 0$ i $x \neq 1$

$t + \frac{1}{t} = \frac{5}{2} \rightarrow$ All multiply with 2t

$$2t^2 + 2 = 5t$$
$$2t^2 - 5t + 2 = 0$$
$$t_{1,2} = \frac{5 \pm 3}{4}$$
$$t_1 = 2$$
$$t_2 = \frac{1}{2}$$

: Back in the replacement : $\log_2 x = 2$, $x = 2^2$, $x = 4$ or $\log_2 x = \frac{1}{2}$, $x = 2^{\frac{1}{2}}$, $x = \sqrt{2}$

4) Solve the equations:

a) $\log_2 x + \log_4 x + \log_{16} x = 7$

b) $\log_3 x \cdot \log_9 x \cdot \log_{27} x \cdot \log_{81} x = \dfrac{2}{3}$

Solution:

In both examples we use that: $\log_{a^S} x = \dfrac{1}{S}\log_a x$

a)

$$\log_2 x + \log_4 x + \log_{16} x = 7 \qquad \text{condition: x>0}$$
$$\log_2 x + \log_{2^2} x + \log_{2^4} x = 7$$
$$\log_2 x + \frac{1}{2}\log_2 x + \frac{1}{4}\log_2 x = 7 \ldots\ldots./\cdot 4$$
$$4\log_2 x + 2\log_2 x + 1\log_2 x = 28$$
$$7\log_2 x = 28$$
$$\log_2 x = 4$$
$$x = 2^4 \Rightarrow x = 16$$

b)

$$\log_3 x \cdot \log_9 x \cdot \log_{27} x \cdot \log_{81} x = \frac{2}{3}$$
$$\log_3 x \cdot \log_{3^2} x \cdot \log_{3^3} x \cdot \log_{3^4} x = \frac{2}{3}$$
$$\log_3 x \cdot \frac{1}{2}\log_3 x \cdot \frac{1}{3}\log_3 x \cdot \frac{1}{4}\log_3 x = \frac{2}{3}$$
$$\frac{1}{24}\log_3^4 x = \frac{2}{3}$$
$$\log_3^4 x = 16 \Rightarrow \log_3 x = t$$
$$t^4 - 16 = 0 \Rightarrow (t^2)^2 - 4^2 = (t^2 - 4)(t^2 + 4) = 0$$
$$(t-2)(t+2)(t^2+4) = 0$$
$$\text{t = 2 or t = -2}$$

$\log_3 x = 2$ or $\log_3 x = -2$

$x = 3^2$ $\qquad$ $x = 3^{-2}$

$x = 9$ or $x = \dfrac{1}{9}$

5) Solve the equations:

a) $\log_{\sqrt{5}}(4^x - 6) - \log_{\sqrt{5}}(2^x - 2) = 2$

b) $\log(7 - 2^x) - \log(5 + 4^x) + \log 7 = 0$

Solution:

a)

$$\log_{\sqrt{5}}(4^x - 6) - \log_{\sqrt{5}}(2^x - 2) = 2$$

$$\log_{\sqrt{5}} \frac{4^x - 6}{2^x - 2} = 2$$

$$\frac{4^x - 6}{2^x - 2} = \sqrt{5}^2$$

$$4^x - 6 = 5(2^x - 2)$$

$$4^x - 6 = 5 \cdot 2^x - 10$$

$$4^x - 5 \cdot 2^x + 4 = 0 \Rightarrow \textit{replacement}: \ 2^x = t$$

$$t^2 - 5t + 4 = 0$$

$$t_{1,2} = \frac{5 \pm 3}{2}$$

$$t_1 = 4$$

$$t_2 = 1$$

$2^x = 4$ or $2^x = 1$

$2^x = 2^2$ $\quad$ $x = 0$

$x = 2$

It is best to check solutions in the home equation → x = 2 is the only solution!

b)

$$\log(7 - 2^x) - \log(5 + 4^x) + \log 7 = 0$$

$$\log_{10}(7 - 2^x) - \log(5 + 4^x) + \log_{10} 7 = 0$$

$$\log_{10}(7 - 2^x) \cdot 7 = \log_{10}(5 + 4^x)$$

$$(7 - 2^x) \cdot 7 = 5 + 4^x$$

$$49 - 7 \cdot 2^x = 5 + 4^x$$

$$49 - 7 \cdot 2^x - 5 - 4^x$$

$$-4^x - 7 \cdot 2^x + 44 = 0 / (-1)$$

$$4^x + 7 \cdot 2^x - 44 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\textit{replacement}: \ 2^x = t$$

$$t^2 + 7t - 44 = 0$$

$$t_{1,2} = \frac{-7 \pm 15}{2}$$

$$t_1 = 4$$

$$t_2 = -11$$

Back in replacement:

$2^x = 4$ Or $2^x = -11$ no solution

$2^x = 2^2$

$x = 2$

Conditions are $7 - 2^x > 0$ and $5 + 4^x > 0$, solution x = 2 obviously satisfy them!

6) Solve the equations:

a) $x^{1+\log x} = 3x$

b) $x^{1+\log_4 x - 2} = 2^{3(\log_4 x - 1)}$

Solution:

a)

$$x^{1+\log x} = 3x................/\log_3$$

$$\log_3 x^{1+\log x} = \log_3 3x \qquad \text{we know that } \log_a b^n = n\log_a b$$

$$(1+\log_3 x)\log_3 x = \log_3 3 + \log_3 x............ \Rightarrow \textit{replacement} \ \log_3 x = t$$

$$(1+t)\cdot t = 1+t$$

$$t + t^2 = 1+t$$

$$t^2 = 1 - t + t \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$$

Back in replacement:

$\log_3 x = 1$ or $\log_3 x = -1$

$x = 3^1$ or $x = 3^{-1}$

$x = 3$ or $x = \frac{1}{3}$

b)

$$x^{1+\log_4 x - 2} = 2^{3(\log_4 x - 1)} \rightarrow \text{log for basis 4 in both sides}$$

$$\log_4 x^{1+\log_4 x - 2} = \log_4 2^{3(\log_4 x - 1)}$$

$$(\log_4 x - 2)\log_4 x = 3(\log_4 x - 1)\log_4 2$$

$$(\log_4 x - 2)\log_4 x = 3(\log_4 x - 1)\log_{2^2} 2$$

$$(\log_4 x - 2)\log_4 x = 3(\log_4 x - 1)\cdot\frac{1}{2}$$

Replacement: $\log_4 x = t$:

$$(t-2)\cdot t=\frac{3}{2}(t-1)$$
$$2t(t-2)=3(t-1)$$
$$2t^2-4t=3t-3$$
$$2t^2-4t-3t+3=0$$
$$2t^2-7t+3=0$$
$$t_{1,2}=\frac{7\pm 5}{4}$$
$$t_1=3$$
$$t_2=\frac{1}{2}$$

So:

$$\log_4 x=3 \quad \text{or} \quad \log_4 x=\frac{1}{2}$$
$$x=4^3 \quad \text{or} \quad x=4^{\frac{1}{2}}$$
$$x=64 \quad \text{or} \quad x=2$$